J. Fluid Mech. (1981), *vol.* 106, *pp.* 27–47
Printed in Great Britain

Some developments in the theory of turbulence

By H. K. MOFFATT
Department of Applied Mathematics and Theoretical Physics,
Silver Street, Cambridge

This is in no way intended as a review of turbulence – the subject is far too big for adequate treatment within a reasonably finite number of pages; the monumental treatise of Monin & Yaglom (1971, 1975) bears witness to this statement. It is rather a discourse on those aspects of the problem of turbulence with which I have myself had contact over the last twenty years or so. My choice of topics therefore has a very personal bias – but that is perhaps consistent with the style and objectives of this rather unusual issue of *JFM*.

I have approached the dynamical problem of turbulence via two simpler (but nevertheless far from trivial) problems – viz the convection and diffusion of a passive scalar field and of a passive vector field by turbulence of known statistical properties. Particular emphasis is given to the method of successive averaging (a simplified version of the renormalization-group technique) which seems to me to have considerable potential. The difficulty of extending this method to the dynamical problem is discussed.

In a final section (§ 6) I have allowed myself the luxury of discussing a somewhat esoteric topic – the problem of inviscid invariants and their relationship with the topological structure of a complex vorticity field. The helicity invariant is the prototype; it is identifiable with the Hopf invariant (1931) and it may be obtained from appropriate manipulation of Noether's theorem (Moreau 1977). A suggestion is made concerning possible measurement of this fundamental measure of 'lack of reflexional symmetry' in a turbulent flow.

1. Preamble

Turbulence is a phenomenon which occurs, whether we like it or not, in an extraordinarily wide range of circumstances, both in technological contexts (e.g. in aerodynamics, hydraulics, naval engineering and chemical engineering), and in the natural contexts provided by geophysics, particularly meteorology and oceanography, and astrophysics. In all such contexts, the Reynolds number R, constructed from scales characterizing the input of energy to the system, is large (typically 10^6 or much greater), and the turbulent state, rather than the laminar state, must be regarded as natural, and unavoidable, in such circumstances.

It is not surprising then that enormous efforts have been devoted on a worldwide scale to fundamental studies of turbulence, and the large number of papers published in *JFM* during the last 25 years on different aspects of the problem of turbulence reflect in some measure the scale and intensity of this activity. There have been striking advances both on the experimental side (notably in the identification of 'coherent structures' – or 'big eddies' as Townsend 1956 called them – in turbulent shear flows) and on the computational side (e.g. in direct numerical simulation of turbulence). On the purely theoretical side, however, fearsome technical difficulties (of which a sample may be found, for example, in papers of Kraichnan 1959, 1965, 1977; Edwards 1964, and others) have been encountered, and the realization has developed, during these last 25 years, that the problem of turbulence, always regarded as difficult, is in fact extremely difficult, and the time-scale of significant advance in understanding has expanded accordingly.

Fundamental studies, such as those mentioned above, are as yet at two stages removed from useful exploitation in technological (or other) contexts. Firstly, they relate generally to the idealization of homogeneous turbulence with zero mean velocity (Batchelor 1953). This idealization, introduced like so many other fundamental ideas in fluid mechanics by G. I. Taylor (1935), has perhaps attracted disproportionate attention by fundamental theoreticians in relation to the immediate applicability of results to turbulent flows of practical importance, in which the mean flow is generally a prominent feature, and which tend to be dominated by interaction between this mean flow and the most energetic ingredients of the turbulence. When I started research myself in turbulence in 1958, I was so preoccupied with understanding and mastering the techniques of homogeneous turbulence that it was fully two years before I became even aware of the fact that shear flow turbulence (boundary layers, pipe and channel flow, jets and wakes and the rest) exists in its own right and requires a somewhat different approach! Curiously, in some respects, the shear flow problem appears to me to be *easier* than the 'homogeneous' problem: if you linearize the shear flow problem (i.e. neglect terms quadratic in fluctuating quantities), you arrive at a non-trivial linear problem (of Orr–Sommerfeld type in plane shear flow) with solutions generally representing damped waves (see, e.g., Landahl 1967), and this at least provides a *starting point* towards understanding the phenomenon of coherent structures and the related problem of energy transfer to the turbulence. On the other hand, if you linearize the homogeneous problem, you are left with the relatively trivial problem of the 'final period of decay' (Batchelor 1953, §5.4) – which by its nature can provide no insight into the central problem of the energy cascade at high Reynolds number. Thus, linearization in shear flow turbulence arguably provides a useful starting point, whereas linearization in homogeneous turbulence does not; in restricting attention to homogeneous turbulence, one is forced to focus attention on those aspects of the problem which are *essentially nonlinear*. Small wonder then that investigations of homogeneous turbulence have been so fraught with difficulty.

The second stage of removal from practicality lies in the fact that even the new insights that are emerging in studies of *shear flow turbulence* are notably difficult to exploit in a useful way in the modelling of turbulence in technological applications. It is perhaps remarkable how persistent the primitive ideas of eddy viscosity, mixing length, and the like, have been in practical contexts where predictions of the effects of turbulence just *have* to be made. It is of course the simplicity of application of these primitive concepts that makes them attractive to users; and if a more sophisticated 'closure' scheme brings no *guarantee* of *better* predictions outside the range of geometries and parameters previously documented, then it is of little value to the user. Sometimes a user may suffer from the delusion that a more 'advanced' closure scheme must, merely by virtue of its complexity, provide a better representation of the effects of turbulence; but close inspection usually reveals that accurate 'prediction', as retrospective rationalization of experimental results is so often misleadingly described, is associated not so much with the genuine physical content of a closure scheme as with the number of disposable parameters that it contains and which may be chosen to optimize the fit with observations. In genuine predictions for complex geometries that have not yet been subjected to experiment (or in extrapolation from laboratory scale to prototype scale) there is an element of chance in the choice of *any* particular closure scheme and, since as yet there is in general no rational basis for the choice, the one that is guided by simplicity and ease of application is surely the one to go for.

Despite this somewhat negative assessment of the current relevance of fundamental studies of homogeneous turbulence to real complex turbulent flows, I remain optimistic that it is simply a matter of time before important bridges are established that *will* permit the incorporation of the results of such studies in practical prediction schemes. If one takes the view that there are features of all turbulent flows which have a universal character (independent of the global geometry and global distribution of energy sources), then it is surely sensible to seek to understand these features in an idealized context in which the global geometry and constraints play a minimal role. Homogeneous turbulence undoubtedly provides one such idealization: the fluid boundaries are imagined removed 'to infinity', and energy may be imagined as 'supplied' to the turbulence (by random stirring) at a rate ϵ per unit mass on some range of length scales of order, say, l_0 and in a statistically uniform manner. The resulting velocity field $\mathbf{u}(\mathbf{x}, t)$ may, on dimensional grounds, be expected† to have a root-mean-square value $u_0 = \langle \mathbf{u}^2 \rangle^{\frac{1}{2}}$ of order

$$u_0 \sim (\epsilon l_0)^{\frac{1}{3}}. \tag{1.1}$$

Its energy spectrum function $E(k)$ may be expected to depend on the spectrum of the stirring forces (or energy input) for $k \sim l_0^{-1}$; but for $k \gg l_0^{-1}$ (i.e. on much smaller scales than l_0) a universal character may reasonably be anticipated.

The dynamics of these small-scale ingredients, and the means by which they extract energy from the 'energy-containing' ingredients ($k \sim l_0^{-1}$), is surely at the heart of the problem of homogeneous turbulence. An understanding of this latter process is crucially important in the problem of 'subgrid-scale modelling' which arises in computational studies of turbulent shear flows of the type initiated by Deardorff (1970); here an attempt is made to follow the detailed evolution of large eddies and energy-containing eddies and their interaction with the mean shear. But finite computer capacity does not permit the same direct numerical simulation of eddies on much smaller scales; at the crudest level, the effect of these smaller eddies may be represented by an eddy viscosity of order $u_g l_g$, where l_g is the scale of the computational grid used, and

$$u_g^2 \sim \int_{l_g^{-1}}^{\infty} E(k)\, dk. \tag{1.2}$$

Subgrid modelling seeks to improve on this primitive description.

The small-scale ingredients are of course also of great importance in their own right, since they play a dominant role in problems of heat and mass transfer by turbulence (see for example the review article, written just 25 years ago, by Batchelor & Townsend

† If, that is, one believes in the irrelevance of viscosity in such estimates when $R \gg 1$.

1956, in G. I. Taylor's 70th anniversary volume). Given the extreme difficulty of the dynamical problem of turbulence, there is much to be said for approaching it via this much easier problem of the evolution of a 'passive' scalar field $\Theta(\mathbf{x},t)$ satisfying the equation

$$\frac{\partial\Theta}{\partial t}+\mathbf{u}\,.\nabla\Theta = \kappa\nabla^2\Theta, \tag{1.3}$$

where κ is the relevant molecular diffusivity, and the statistical properties of $\mathbf{u}(\mathbf{x},t)$ are (for the moment) assumed known. I shall discuss some aspects of this problem in the following section.

An even better springboard for the study of the dynamical problem is provided by study of the evolution of a passive solenoidal *vector* field $\mathbf{B}(\mathbf{x},t)$ satisfying the equation

$$\frac{\partial\mathbf{B}}{\partial t} = \nabla\wedge(\mathbf{u}\wedge\mathbf{B})+\eta\nabla^2\mathbf{B}, \tag{1.4}$$

where again the statistical properties of $\mathbf{u}$ are assumed known. (The magnetic field $\mathbf{B}$ in a fluid of magnetic diffusivity η satisfies just this equation.) The formal similarity between (1.4) and the vorticity equation (with zero forcing),

$$\frac{\partial\boldsymbol{\omega}}{\partial t} = \nabla\wedge(\mathbf{u}\wedge\boldsymbol{\omega})+\nu\nabla^2\boldsymbol{\omega}, \tag{1.5}$$

was pointed out by Elsasser (1946) and developed in the context of turbulence by Batchelor (1950). The supplementary constraint, $\boldsymbol{\omega} = \nabla\wedge\mathbf{u}$, of course makes (1.5) non-linear and profoundly difficult to handle. In (1.4), $\mathbf{B}$ is *freed* from this constraint and a *wider* class of problems is therefore covered by (1.4) than by (1.5). If generality brings linearity as a bonus, no further motivation is needed for the detailed study of (1.4). However, enthusiasm for this procedure should be tempered by the following consideration: an explicit solution of (1.4) of the form

$$\mathbf{B}(\mathbf{x},t) = \mathscr{F}\{\mathbf{u}(\boldsymbol{\xi},\tau);\ \mathbf{x},t;\eta\}, \tag{1.6}$$

where the functional on the right depends on values of $\mathbf{u}$ at all points $\boldsymbol{\xi}$ and all times $\tau < t$, would not in fact provide a solution of (1.5); the formal statement analogous to (1.6) is

$$\boldsymbol{\omega}(\mathbf{x},t) = \mathscr{F}\{\mathbf{u}(\boldsymbol{\xi},\tau);\ \mathbf{x},t;\ \nu\}, \tag{1.7}$$

which, in conjunction with

$$\mathbf{u}(\boldsymbol{\xi},\tau) = \frac{1}{4\pi}\int\frac{(\boldsymbol{\xi}-\boldsymbol{\xi}')\wedge\boldsymbol{\omega}(\boldsymbol{\xi}',\tau)}{|\boldsymbol{\xi}-\boldsymbol{\xi}'|^3}+\text{surface contributions}, \tag{1.8}$$

would merely provide a restatement of (1.5) as an integral equation. Despite this reservation, *some* results relating to the $\mathbf{B}$ field *can* usefully be carried over to the $\boldsymbol{\omega}$ field, and I am of the view that the analogy between (1.4) and (1.5) is a powerful one that has not yet been exploited to the fullest possible extent.

2. The scalar field problem

One of Taylor's great contributions to the theory of turbulence lay in his (1921) description of the dispersion of marked fluid particles by turbulence. This description leads to the asymptotic law

$$\langle \mathbf{X}^2 \rangle \sim 2D_m t, \tag{2.1}$$

where $\mathbf{X}(\mathbf{a}, t)$ is the position of the marked particle at time t, labelled by its initial position $\mathbf{a}$ (with $\langle \mathbf{X} \rangle = 0$, the mean velocity being assumed zero). D_m is the effective diffusivity, and, in isotropic turbulence, it is given by

$$D_m = \frac{1}{3}\int_0^\infty \langle \mathbf{v}(\mathbf{a}, t) . \mathbf{v}(\mathbf{a}, t+\tau) \rangle \, d\tau, \tag{2.2}$$

where $\mathbf{v} = \partial \mathbf{X}/\partial t$ is the Lagrangian velocity. The appearance of the velocity correlation function in (2.2) is highly significant; the Taylor description of turbulent diffusion constituted the first genuinely statistical approach to turbulence; it was no doubt this natural appearance of a correlation function which led to his subsequent (1935) formulation of the dynamical problem in terms of (Eulerian) correlation functions. The problem of expressing Lagrangian correlations in terms of Eulerian correlations remains largely unsolved to this day – for a discussion of great interest in this context, see Patterson & Corrsin (1966).

The result (2.2) can be obtained directly from (1.3), assuming $\kappa = 0$ (as is appropriate for a field of 'marked particles' which do not lose their identity as they follow the fluid motion). It is therefore tempting to suppose (as is often done) that (2.2) provides the correct expression for the effective eddy diffusivity of turbulence acting on a scalar field of scale L large compared with the turbulent scale l_0, in the asymptotic limit

$$Pe = \frac{u_0 l_0}{\kappa} \to \infty. \tag{2.3}$$

There is, however, a fundamental difficulty associated with the total neglect of molecular diffusivity effects. It is clear that continuous mixing of a non-uniform Θ field with $\kappa = 0$ may lead to unbounded increase in $|\nabla\Theta|$; and in fact, writing $\theta = \Theta - \langle \Theta \rangle$, it is easily shown from (1.3) that, when $\kappa = 0$, $\langle (\nabla\theta)^2 \rangle$ does indeed in general increase without limit under the action of turbulence. In a fluid with non-zero κ (no matter how small) this result is clearly unphysical. The actual magnitude of $\langle (\nabla\theta)^2 \rangle$ under statistically steady conditions is determined by the fact (Batchelor 1959) that $\kappa\langle (\nabla\theta)^2 \rangle$ remains independent of κ in the limit $\kappa \to 0$; in dimensionless terms,

$$\langle (\nabla\theta)^2 \rangle \sim Pe(\nabla\Theta_0)^2, \tag{2.4}$$

where $\Theta_0 = \langle \Theta \rangle$.

Since it is necessary to invoke molecular diffusivity effects in order to arrive at a finite value for $\langle (\nabla\theta)^2 \rangle$, it is arguable that the same molecular diffusivity effects may play an important role in determining the effective diffusivity D acting on a dispersed cloud of passive contaminant. This 'interaction' problem was addressed by Saffman

(1960, 1962) who showed that, if a heat spot is released at $t = 0$ in a turbulent flow, the dispersion for small t is given by

$$2\left(D_m t + \kappa t - \frac{\kappa u_0^2}{\nu t_L} t^3 + O(t^4)\right), \tag{2.5}$$

where D_m is the Taylor diffusivity, as given above, and t_L is the Lagrangian correlation time. The reduction in dispersion (at order t^3) due to interaction between convective and diffusive effects was noteworthy, and led Saffman, via intuitive arguments, to the conclusion that a similar reduction would persist asymptotically for *large* times; indeed, from Saffman's results, one may infer an asymptotic effective diffusivity

$$D_e = D_m + \kappa - CR_\lambda \kappa, \tag{2.6}$$

where $C = O(1)$, $R_\lambda = u_0\lambda/\nu$ and $\lambda = (15u_0^2/\langle\omega^2\rangle)^{\frac{1}{2}}$ is the dissipation length scale. The reduction term may be large, even if κ is small, when $R_\lambda \gg 1$. Arguments supporting Saffman's conclusion have been developed by Phythian & Curtis (1978).

Saffman's approach was essentially Lagrangian in character, and it is mathematically secure only on the short time scales for which (2.5) is valid. An alternative approach, which is Eulerian in character, has been advocated by Rose (1977) in a 'subgrid-scale modelling' study, and seems to me to offer great promise. The idea, derived from the 'renormalization-group' techniques of quantum field theory (Nelkin 1974, 1975), may be explained in crude and simple terms as follows.† Suppose we imagine the $\mathbf{u}$ and θ fields decomposed into ingredients $(\mathbf{u}_r, \theta_r)$ $(r = 1, 2, \ldots, n)$ on scales l_r with

$$l_n \lesssim l_{n-1} \lesssim \ldots \lesssim l_1. \tag{2.7}$$

(Equivalently, wavenumber space may be considered decomposed into spherical shells Δ_r: $k_r < k < k_{r+1}$ with $k_r = l_r^{-1}$.) The smallest scale l_n can be taken as the scale of the conduction cut-off (Batchelor 1959); for $\kappa \gtrsim \nu$, this is

$$l_n \sim (\kappa^3/\epsilon)^{\frac{1}{4}}, \tag{2.8}$$

where ϵ is as previously defined. The Péclet number Pe_n based on u_n and l_n is then (at most) of order unity, and the eddy diffusivity associated with the ingredient $\mathbf{u}_n$ (assumed isotropic) is then given by

$$D_n \approx \frac{2}{3\kappa}\int_{\Delta_n} k^{-2}E(k)\,dk \quad (\sim Pe_n^2\,\kappa) \tag{2.9}$$

where Δ_n is the 'shell' $k > k_n$. This expression may be obtained by standard methods;‡ strictly it is valid only if $Pe_n \ll 1$, although it may also be reasonably accurate for all $Pe_n \lesssim 1$.

The effective diffusivity acting upon $\theta_1 + \ldots + \theta_{n-1}$ is then $\kappa + D_n$; averaging over the scale l_n has led to a small increase in the diffusivity (in technical language, it has

† The essence of the argument presented here, and the result (2.12), are due to Howells (1960) – one of the shortest papers ever published in *JFM*!

‡ This involves calculation of θ_n in terms of $\mathbf{u}_n$ from the diffusion equation $\kappa\nabla^2\theta_n \approx \mathbf{u}_n.\nabla\Theta_n$ $(\Theta_n = \theta_1 + \theta_2 + \ldots + \theta_{n-1})$; then construction of $\langle\mathbf{u}_n\theta_n\rangle = -D_n\nabla\Theta_n$.

been renormalized). We may now repeat the process, averaging now over the larger scale l_{n-1}; we then find an incremental diffusivity

$$D_{n-1} \approx \frac{2}{3(\kappa + D_n)} \int_{\Delta_{n-1}} k^{-2} E(k)\, dk, \tag{2.10}$$

and so on.

If we now let $n \to \infty$ and each $\Delta_n \to 0$, then, writing $D(k)$ for the eddy diffusivity acting on all scales $< k^{-1}$, relations of the type (2.9), (2.10), ... are all replaced by the differential relationship

$$dD = -\frac{2}{3(D+\kappa)} k^{-2} E(k)\, dk, \tag{2.11}$$

which integrates very easily (with boundary condition $D(\infty) = 0$) to give

$$(D+\kappa)^2 = \frac{4}{3}\int_k^\infty k_1^{-2} E(k_1)\, dk_1 + \kappa^2. \tag{2.12}$$

The first term on the right-hand side dominates when $k \sim l_0^{-1}$, and the effective diffusivity acting on scales $L \gtrsim l_0$ (i.e. acting on the mean field Θ_0) is then

$$D_0 = \left\{\frac{4}{3}\int_{k_0}^\infty k^{-2} E(k)\, dk\right\}^{\frac{1}{2}}. \tag{2.13}$$

This expression (which contains no undetermined dimensionless constants) is to be compared with the expression (2.2). In order of magnitude (to within a factor of order unity)

$$D_0 \sim \tfrac{1}{3} l_0 u_0, \quad D_m \sim \tfrac{1}{3} u_0^2 t_L, \tag{2.14}$$

where now l_0 is the Eulerian integral scale. The experimental results of Snyder & Lumley (1971) show that $t_L \approx l_0/u_0$, so that D_0 is of the same order of magnitude as D_m. Thus inclusion of the effects of molecular diffusivity has apparently little effect in modifying the Taylor diffusivity. Experimental discrimination between the result (2.12) and Saffman's prediction (2.6) appears desirable. The laser technique developed by Fermigier (1980) may be well adapted for this type of investigation.

A result of the form (2.12) implies a simple relationship between the constants C and B occurring in the inertial range laws

$$E(k) = C\epsilon^{\frac{2}{3}} k^{-\frac{5}{3}}, \quad \Gamma(k) = B\eta\epsilon^{-\frac{1}{3}} k^{-\frac{5}{3}}, \tag{2.15}$$

where $\Gamma(k)$ is the spectrum function of the θ field, and η is the rate of cascade of $\langle\theta^2\rangle$-stuff (Batchelor 1959) through the spectrum. For consistency, we must have†

$$\eta = D(k)\int_0^k k^2\, \Gamma(k)\, dk = \kappa \int_0^\infty k^2\, \Gamma(k)\, dk. \tag{2.16}$$

In the part of the inertial range for which both of (2.15) are valid,

$$D(k) = \left[\frac{4}{3}\int_k^\infty E(k)\, k^{-2} dk\right]^{\frac{1}{2}} \sim \left(\frac{C}{2}\right)^{\frac{1}{2}} \epsilon^{\frac{1}{3}} k^{-\frac{4}{3}}, \tag{2.17}$$

and

$$\int_0^k k^2 \Gamma(k)\, dk \sim \tfrac{3}{4} B\eta\epsilon^{-\frac{1}{3}} k^{\frac{4}{3}}, \tag{2.18}$$

† This neglects transfer of $\langle\theta^2\rangle$-stuff across wavenumber k by the straining action of larger eddies – see Howells (1960).

and hence, from (2.16),

$$B = \frac{4}{3}\left(\frac{2}{C}\right)^{\frac{1}{2}}. \tag{2.19}$$

The constant C is (experimentally) better determined than B, and the most reliable value is given by Monin & Yaglom (1975, p. 485) as $C \doteq 1{\cdot}5$. The corresponding value of B from (2.19) is also $B \approx 1{\cdot}5$. The experimental evidence collected by Monin & Yaglom (1975, pp. 497–505) suggests a preferred value $B \approx 1{\cdot}4$, but the scatter is considerable. The result (2.19) is therefore certainly not inconsistent with observation.

The above calculation is open to criticism on a number of grounds, but I have thought it useful to include it in the present discussion, because it seems to me to demonstrate the potential power and essential simplicity of the 'successive averaging' or 'renormalization-group' technique. I have deliberately simplified the discussion here, but readers who have not done so already may be encouraged to study the paper of Rose (1977) in which the treatment is more precise and the effects explored more subtle. The renormalization-group technique has been systematically expounded in the context of turbulence by Forster, Nelson & Stephen (1977) but, as indicated by the title of their article, 'Large distance and long-time properties of a randomly stirred fluid', the aim there was to understand asymptotic large-scale behaviour ($k \to 0$ in spectral terminology, cf. Batchelor & Proudman 1956; Saffman 1967) rather than properties of the energy-containing ingredients of the turbulence.

3. The vector field problem; helicity and the α-effect

The double-length scale approach was introduced to marvellous effect in the context of the equation

$$\partial \mathbf{B}/\partial t = \nabla \wedge (\mathbf{u} \wedge \mathbf{B}) + \eta \nabla^2 \mathbf{B}, \quad \nabla . \mathbf{B} = 0, \tag{3.1}$$

by Steenbeck, Krause & Rädler (1966). This work is perhaps chiefly of interest in geomagnetic and astromagnetic contexts; but, as I have argued in § 1, an understanding of the consequences of (3.1) is a valuable preliminary to any attack on equation (1.5), and it is in this spirit that I shall discuss the topic here.

The above paper by Steenbeck, Krause & Rädler, and the series of papers which followed in the period 1966–70, were published in German, and it was some time before the results filtered through to the West.† I first learnt of the existence of the papers in 1969, and was acutely interested to read them, as I was working on the same problem at that time. Never was I more conscious of my total lack of knowledge of the German language! A young lecturer in DAMTP of German nationality came to my rescue, and went over the papers with me line by line. The outstanding discovery, recorded in the 1966 paper, was the vital relevance of 'Schraubensinn', literally 'screw-sense', now better known as 'helicity', in the statistical analysis of (3.1). Defining the mean helicity of a homogeneous field of turbulence as

$$\mathscr{H} = \langle \mathbf{u} . \boldsymbol{\omega} \rangle, \tag{3.2}$$

† Paul Roberts and Michael Stix provided an invaluable service when they published an English translation of the papers as NCAR-TN-1A-60 in June 1971.

what Steenbeck & Krause (1966) had in effect shown was that, if $\mathscr{H} \neq 0$, then, in general, equation (3.1) admits instabilities on scales large compared with the turbulent scale l_0. The implications in geomagnetic and astromagnetic contexts have been immense. The recent publication of the theory in book form (Krause & Rädler 1981) is greatly to be welcomed; for complementary accounts of this rapidly growing field of activity, see also Moffatt (1978) and Parker (1979).

The essence of the two-scale approach is as follows: let $\mathbf{u} = \mathbf{U}+\mathbf{u}'$, $\mathbf{B} = \mathbf{B}_0+\mathbf{b}'$, where $\langle \mathbf{u}' \rangle = \langle \mathbf{b}' \rangle = 0$. Then the mean of (3.1) is

$$\partial \mathbf{B}_0/\partial t = \nabla \wedge \boldsymbol{\mathscr{E}} + \nabla \wedge (\mathbf{U} \wedge \mathbf{B}_0) + \eta \nabla^2 \mathbf{B}_0, \tag{3.3}$$

where $\boldsymbol{\mathscr{E}} = \langle \mathbf{u}' \wedge \mathbf{b}' \rangle$. The fluctuation equation for $\mathbf{b}'$ determines $\mathbf{b}'$ (and hence $\boldsymbol{\mathscr{E}}$) as a linear functional of $\mathbf{B}_0$. In the simplest case of isotropic turbulence, the relation between $\boldsymbol{\mathscr{E}}$ and $\mathbf{B}_0$ takes the form

$$\boldsymbol{\mathscr{E}} = \alpha \mathbf{B}_0 - \beta \nabla \wedge \mathbf{B}_0 + \dots, \tag{3.4}$$

where ... indicates terms involving higher derivatives of $\mathbf{B}_0$, which may be expected to be small when the scale of the mean field $\mathbf{B}_0$ is sufficiently large. Substitution in (3.3) gives

$$\partial \mathbf{B}_0/\partial t = \nabla \wedge (\mathbf{U} \wedge \mathbf{B}_0) + \alpha \nabla \wedge \mathbf{B}_0 + (\eta+\beta) \nabla^2 \mathbf{B}_0. \tag{3.5}$$

Here α is a pseudo-scalar which is non-zero only if the turbulence *lacks reflectional symmetry* – in which case $\mathscr{H}$ is in general non-zero also; β on the other hand is a pure scalar, and it clearly plays the role of an eddy diffusivity. Equation (3.5) is known to admit unstable solutions for many geometrical configurations and many choices of $\mathbf{U}(\mathbf{x})$, α and $\eta+\beta$ — see the three books cited above.

The parameters α and β now have to be determined and this problem is analogous to the problem of determining the eddy diffusivity D for a scalar field (§2). The difficulties already encountered in the weak diffusion limit (here $\eta \to 0$) now become more acute. Expressions analogous to Taylor's expression (2.2) can be obtained for α and β by Lagrangian analysis, and these expressions have been evaluated in numerical simulation experiments by Kraichnan (1976*a*, *b*); but the same analysis implies unlimited increase in $\langle \mathbf{b}'^2 \rangle$ which is clearly unphysical. Only restoration of molecular diffusivity effects can yield a physically sensible result for this quantity.

This problem is of central interest in its own right in astrophysical contexts. For example, the magnetic Reynolds number $R_m = u_0 l_0/\eta$ associated with turbulence in the solar convection zone is of order 10^4 or 10^5 (and $R \sim 10^7 R_m$), and there is no escape from consideration of the weak diffusion limit ($\eta \ll u_0 l_0$).

The problem is also of acute interest in relation to its possible relevance to the analogous problem presented by the dynamical equation (1.5) when

$$R = u_0 l_0/\nu \gg 1:$$

if we cannot handle (3.1) when $R_m \gg 1$, what hope have we of ever being able to handle (1.5) when $R \gg 1$?

The question therefore arises as to whether the renormalization-group technique, described in §2, can be adapted to the vector field problem, with a view to determining asymptotic expressions for α and β as $R_m \to \infty$. The formalism of Forster *et al.* (1977) has in fact been extended to hydromagnetic turbulence by Fournier (1977) (see also

Pouquet, Fournier & Sulem 1978), but again with a view to asymptotic analysis of large-scale structures. A simple-minded approach (in the style of § 2) runs immediately into an *interesting* difficulty, in that the averaged equation (3.5) does *not* have the same mathematical structure as the parent equation (3.1) – it is of course this change of structure which makes the α-effect (as the appearance of the term $\alpha\mathbf{B}_0$ in (3.4) is known) of such fundamental importance; but, at the same time, it means that the problem (3.1) is not renormalizable in the same straightforward sense as was the problem (1.3).

However, in the case of reflexionally symmetric turbulence, which is of course not without interest, $\alpha = 0$, and (3.5) *does* then have the same structure as (3.1), but with an augmented (renormalized) diffusivity. In this case, a successive averaging procedure can be set up, exactly analogous to that described in § 2, and with the same conclusion, viz that, when $R_m \gg 1$, the effective eddy diffusivity $\beta(k)$ acting on all scales larger than k^{-1} is given by

$$\beta(k) = \left\{\frac{4}{3}\int_k^\infty k_1^{-2} E(k_1)\, dk_1\right\}^{\frac{1}{2}} \tag{3.6}$$

(this result holding only for the range of k for which $\beta(k) \gg \eta$). If the renormalization-group procedure can be rigorously justified, we thus conclude that, in reflexionally symmetric turbulence, the eddy diffusivity acting upon a passive vector field is exactly equal to that acting upon a passive scalar field. This is reminiscent of the claim of Parker (1971) who argued (from a Lagrangian standpoint) that both diffusivities are given by Taylor's formula (2.2). The argument based on successive averaging is totally different, but the conclusion $\beta = D$ is the same: *plus ça change, plus c'est la même chose*! But, of course, the expression (3.6) is *not* the same as the expression (2.2).

When the turbulence is not reflexionally symmetric, then, as observed above, equation (3.5) has a different structure from (3.1), and an interesting departure from the scalar field problem is to be expected. The same change in structure of course appears in the multiple-scale approach of § 2, in averaging over the innermost scale l_n. However a second averaging process over the scale l_{n-1} does not lead to any *further* change of structure. This means that the successive averaging process *can* be established (as, in effect, found also by Fournier 1977), and transition to the limit $n \to \infty$, $\Delta_n \to 0$ (as in § 2) leads to coupled differential equations for $\alpha(k)$ and $\beta(k)$, somewhat analogous to (2.11). It would be inappropriate to go into the details here, but one important conclusion (as found by Kraichnan 1976*a* – see also Moffatt 1978, § 7.11) is worth stating: the presence of an α-effect on small scales leads to a *decrease* in the effective diffusivity operating at larger scales. Kraichnan (1978) has argued that helicity fluctuations that are sufficiently extensive and persistent can even result in a *negative* total effective diffusivity – a conclusion that carries dramatic consequences as far as the (largest-scale) mean magnetic field is concerned.

4. Vorticity dynamics and the energy cascade

In 1941, Kolmogorov presented his seminal paper 'The local structure of turbulence in incompressible viscous fluid for very large Reynolds numbers' in the *Comptes Rendus* (*Doklady*) *de l'Académie des Sciences de l'URSS*. With characteristic brevity, he introduced the concept of local isotropy, and his two similarity hypotheses, and he deduced that the mean square of the velocity difference at two points separated by a distance r is proportional to $r^{\frac{2}{3}}$, provided $(\nu^3/\epsilon)^{\frac{1}{4}} \ll r \ll l_0$. The corresponding spectral statement (Batchelor 1953) is

$$E(k) = C\epsilon^{\frac{2}{3}} k^{-\frac{5}{3}} \quad (k_0 \ll k \ll (\epsilon/\nu^3)^{\frac{1}{4}}). \tag{4.1}$$

Kolmogorov's theory of the local structure of turbulence was beautiful in its universality, its simplicity, and its immediate applicability to problems (such as the break-up of small droplets by turbulence – Kolmogorov 1949) in which the small-scale ingredients of the turbulence play a dominant role. Experimental results supporting the theory, and in particular the central conclusion (4.1), were therefore keenly sought during the 1950s, particularly in studies of atmospheric turbulence (Gurvich 1960). The difficulty was to find a field of turbulence at sufficiently high Reynolds number to guarantee an 'inertial range' of wavenumbers of sufficient extent for the result (4.1) to be convincingly demonstrated.

In 1961, an important Colloquium on turbulence was held at Marseille, on the occasion of the opening of the Institut de Mécanique Statistique de la Turbulence. It was my first experience of international meetings, and I approached it with a considerable degree of excitement and expectation. Von Kármán was there, and so were Kolmogorov and G. I. Taylor. I recall that von Kármán, in his opening address, said that, when he finally came face to face with his Creator, the first revelation he would supplicate would be an unfolding of the mysteries of turbulence.† Certain other events of the Marseille meeting stand out in my memory – and among these, the drama over the $k^{-\frac{5}{3}}$-law was dominant. The experimental evidence presented at the meeting by Bob Stewart (subsequently published in *JFM* – Grant, Stewart & Moilliet 1962) appeared to clinch matters: these experiments, conducted at a Reynolds number of 3×10^8 in the tidal channel between Vancouver Island and mainland Canada, provided convincing support for the $k^{-\frac{5}{3}}$-law, over several octaves, the value of C inferred from the experiments being

$$C = 1{\cdot}44 \pm 0{\cdot}06. \tag{4.2}$$

So there it was: a classic example of long-awaited experimental evidence providing confirmation of a theoretical argument of central importance.

And yet there was a serious problem, which was just beginning to surface at that time, a problem that was to seriously affect the credibility of the Kolmogorov (1941) theory; this was the problem of intermittency, discussed in §5 of the paper by Grant *et al.* cited above. And indeed, at the same Marseille meeting, Kolmogorov himself turned his attention to this problem – his paper is published (in English)‡ in *JFM* (1962) volume 13 (see also the closely related paper by Oboukov 1962 in the same volume). This was one of the first papers which I, as a new recruit to the *JFM* editorial team, prepared for the Press – and it was not an easy assignment! By taking account

† A similar sentiment is attributed to Horace Lamb, in the review 'One hundred years of Lamb's *Hydrodynamics*' by L. Howarth (*J. Fluid Mech.* vol. 90, pp. 202–207).

‡ It was later published in both French and Russian in the *Proceedings* of the Meeting.

of intermittent spatial fluctuations in the rate of dissipation ϵ, Kolmogorov showed that (4.1) should be replaced by

$$E(k) = C\epsilon^{\frac{2}{3}} k^{-\frac{5}{3}} (kl_0)^{-\delta}, \tag{4.3}$$

where δ is a small positive number. The same δ appears in expressions for higher-order spectral quantities, which are more sensitive to intermittency effects – see Monin & Yaglom 1975, §25, where the estimate $\delta \approx 0{\cdot}055$ is given. The resulting modification of (4.3) was slight; the modification of the underlying similarity hypotheses was nevertheless profound. Gone was the beautiful simplicity of the earlier theory; from 1961 on, *no* aspect of turbulence was to be 'simple'.

I have mentioned the renormalization-group technique, as developed by Forster *et al.* (1977) in the dynamical context (see also Rose & Sulem 1978; Frisch, Sulem & Nelkin 1978). Following the more simple-minded 'successive averaging' approach described in §§2 and 3, one might now hope to exploit the analogy between (3.1) and (1.5) to obtain useful results. Here, one would take the view that small-scale vorticity is generated by convection and distortion of large-scale vorticity by small-scale velocity – and build up a successive averaging procedure just as in §2. Despite the analogy, however, the procedure (in this simple form) does not work, and it is interesting to see why. If we consider a single realization of a turbulent flow containing only two length scales l_1 and l_2 ($l_1 \gg l_2$), and we average the vorticity equation (1.5) over the inner scale l_2, then we have to calculate

$$\langle \mathbf{u}_2 \wedge \boldsymbol{\omega}_2 \rangle = \nabla . \langle \mathbf{u}_2 \mathbf{u}_2 - \tfrac{1}{2} u_2^2 \mathbf{I} \rangle, \tag{4.4}$$

where $\mathbf{I}$ is the unit dyadic. This expression is non-zero only if the statistical properties of $\mathbf{u}_2$ are *inhomogeneous*. Such inhomogeneity will of course be induced by the non-uniform straining action of the larger scale field $\mathbf{u}_1$ – but this is an effect that plays a negligible part in determining $\langle \mathbf{u}_2 \theta_2 \rangle$ and $\langle \mathbf{u}_2 \wedge \mathbf{b}_2 \rangle$ in the passive scalar and vector field problems. The vector field $\boldsymbol{\omega}_2$ is *not* passive – it 'reacts back', via the relation $\boldsymbol{\omega}_2 = \nabla \wedge \mathbf{u}_2$, on the velocity field $\mathbf{u}_2$ that generates it. This makes the dynamic problem much more subtle than the passive vector field problem – and it would perhaps be pushing luck too far simply to carry over the eddy diffusivity $\beta(k)$ given by (3.5) to the dynamic context (and to relabel it 'eddy viscosity'). This would be reminiscent of Heisenberg's (1948) expression

$$\beta(k) = \gamma \int_k^\infty k^{-\frac{3}{2}} [E(k_1)]^{\frac{1}{2}} dk_1, \tag{4.5}$$

where γ is a constant of order unity – and a theory based on it would run into well-known difficulties at large wavenumbers (see Batchelor 1953, §7.5).

The need to take account of the action of large-scale straining on small-scale vorticity was what motivated Pearson (1959) to solve this problem on the basis of rapid distortion theory; this produced the surprising result that $\langle \boldsymbol{\omega}_2^2 \rangle$ in general increases without limit despite the action of viscosity. As argued by Monin & Yaglom (1975, §22.3) this infinity can be avoided by filtering out 'irrelevant' large-scale contributions to Pearson's vorticity field; this approach was in fact used by Novikov (1961) to yield an expression for the spectrum function in the far dissipation range

$$E(k) \sim C\epsilon^{\frac{2}{3}} k^{-\frac{5}{3}} (kl_v)^{2\sigma-\frac{4}{3}} e^{-\alpha(kl)_v^2}, \tag{4.6}$$

where l_v is the Kolmogorov inner scale, and σ and α are statistical parameters associated with the strain field. The approach is an attractive one, which could possibly provide an alternative starting point for the renormalization-group procedure.

The phenomenon of intermittency has stimulated some imaginative approaches to the kinematics of turbulence – among which Mandelbrot's (1974, 1975) description in terms of fractional dimension deserves special mention. If we permit ourselves for the moment to imagine turbulence in the limit $\nu \to 0$, then the question arises as to what is the limiting structure of the vorticity field in real space. The indications are that, if we start from some random smooth initial condition, then $\langle \boldsymbol{\omega}^2 \rangle$ will in general develop a singularity in a finite time, of order l_0/u_0, as conjectured by Brissaud *et al.* (1973).† The $k^{-\frac{5}{3}}$-spectrum (or something very close to it) will then extend to $k = \infty$; the time t_k associated with eddies of scale k^{-1} is then (on dimensional grounds)

$$t_k \sim \epsilon^{-\frac{1}{3}} k^{-\frac{2}{3}},$$

and this is also the time characteristic of energy transfer from scale k^{-1} to scale $(2k)^{-1}$; energy therefore cascades from k_0 to $k = \infty$ in a total time of order

$$T_0 = \int_{k_0}^{\infty} t_k \, d(\ln k) \sim \epsilon^{-\frac{1}{3}} k_0^{-\frac{2}{3}} \sim l_0/u_0, \tag{4.7}$$

reflecting the 'finite-time' singularity. A k^{-2}-spectrum for the velocity field would reflect a physical structure having a finite number of discontinuities per unit length in any direction (e.g. the vortex sheet and line model of Townsend 1951 has this property). The *slower* fall-off described by the $k^{-\frac{5}{3}}$-spectrum suggests a (mildly) *worse* physical structure – e.g. a situation in which vortex sheets are in some regions infinitely convoluted. Mandelbrot (1975) conceives of these surfaces as being so convoluted (in the limit $\nu \to 0$) as to occupy a space of (Hausdorff) dimension intermediate between 2 and 3 — an appealing idea, which however is difficult to incorporate in dynamical arguments based on the Navier–Stokes equation.

In this regard, a phenomenon of the 70s has been the increasing awareness of 'pure' mathematicians (i.e. those for whom rigour has top priority) of the rich fascination of the Navier–Stokes equations and related nonlinear systems. The range of pure techniques that have been brought to bear on the problem of turbulence is admirably represented in *Turbulence and Navier–Stokes Equations* [Lecture Notes in Mathematics (ed. R. Temam) no. 565, Springer, 1976]. In reading this volume, one cannot fail to be struck by the contrast in linguistic style between the papers that it contains and 'typical' papers on turbulence published in *JFM* (e.g. those to which I have referred in this article). As an example, I quote a theorem of Scheffer (1976) (motivated by the ideas of Mandelbrot concerning the relevance of 'fractal' dimension): 'Let u be a "solution turbulente" with finite initial kinetic energy such that the initial conditions are smooth. Let $T > 0$ be given and set $A = \{x \in R^3$: the restriction of u to $\{x\} \times ([0, T] \cap (U_{q \geqslant 0} J_q))$ is a bounded function$\}$. Then the Hausdorff dimension of $R^3 - A$ is at most 5/2.' How many readers of *JFM* could claim to feel at home with language such as this? There is a danger of a severe communications barrier between 'purists' and 'traditionalists' arising from insufficient attempt on either side to express problems or results in terms that the other can comprehend. A new language barrier appears to be developing here and, unfortunately, interpreters are as yet few. In this context, purists might do well to note that all significant advances in turbulence

† In this context, see the interesting new developments discussed by Saffman, in this volume.

theory in the past have been guided by powerful physical reasoning; mathematical techniques in isolation have had little to offer – except toil and tribulation!

5. Some comments on the 'rapid distortion' approach to shear flow turbulence

The first edition of Townsend's monograph *The Structure of Turbulent Shear Flow* was published in 1956, and the second edition in 1976. An interesting insight into the changing ideas of the intervening period may be obtained from close comparison of the two editions. In particular, the whole of chapter 4, concerned in the original edition with 'Uniform distortion of homogeneous turbulence' was rewritten, and retitled 'Inhomogeneous shear flow'. In 1956, the emphasis had been on the distortion of turbulence by uniform irrotational strain, whereas in 1976 the emphasis was shifted to the problem of interaction of turbulence with uniform shear flow. Townsend here reproduced the calculations of his 1970 *JFM* paper in which he showed, remarkably, that a finite shear of about 63° imposed on initially isotropic turbulence, can reproduce quite accurately eight out of the nine principal correlation functions measured by Grant (1958) in a turbulent wake – and this all on the basis of a linear calculation! The relevance of the uniform shear 'rapid distortion' calculation, and the irrelevance of the earlier irrotational strain calculations in shear flow contexts, were simultaneously established. I may say that I gained some satisfaction myself from this considerable change of emphasis, having pressed the same point of view somewhat earlier (Moffatt 1967*a*) in the context of atmospheric turbulence with strong wind shear.

As Deissler (1961) had shown even earlier, a crucially important effect of mean shear acting on initially isotropic turbulence is the selective amplification of structures having large length scale in the mean flow direction – a property that reappeared in the very slow fall-off of Townsend's correlation curves when the separation vector $\mathbf{r}$ is in the mean flow direction.

Since 1970, rapid distortion theory has received a new lease of life, largely stimulated by the pioneering study of Hunt (1973) of the effects of turbulence in a stream incident on large bluff bodies. Here, irrotational distortion *is* relevant, as is the blocking effect of the solid boundary on large eddies. These effects have been encountered in numerous subsequent studies (see, e.g., Hunt & Graham 1977; Hunt 1978; Britter, Hunt & Mumford 1979). The success of rapid distortion theory, in which nonlinear interactions between turbulent fluctuations are neglected for the duration of the distortion, again illustrates that, in *some* respects, shear flow turbulence can be *easier* than homogeneous turbulence (with zero mean flow), in that a linearized model can at least provide a useful starting point.

It is important to recognize that rapid distortion theory works best when a turbulent flow is subjected to a *brutal* change; it was originally devised to describe the effects of a contraction in a wind tunnel; similarly, it might be expected to work well for turbulent flow round a sharp and substantial bend in a channel, in relating the statistical properties of the turbulence immediately after the bend to those immediately before the bend.

Linearized theory is equally valid under other brutal changes in external conditions. For example, if a turbulent flow of a liquid metal is subjected to the sudden application of a strong magnetic field, it relaxes during an initial linear phase to a strongly anisotropic (indeed nearly two-dimensional) structure (Moffatt 1967*b*; Alemany *et al.* 1979).

6. Helicity and the topological structure of the vorticity field

I would like to make some comments in this section on a topic that appears to me to be of fundamental interest in fluid mechanics generally, and also to have important implications for turbulence (some of which have been explored, for example, by Pouquet, Frisch & Léorat 1976). It was of course known to Kelvin (1868) that, when $\nu = 0, p = p(\rho)$ and body forces (per unit mass) are conservative, vortex lines are frozen in the fluid, and that in consequence knots and linkages in vortex lines are inevitably conserved. I wrote a paper (Moffatt 1969) showing that the helicity of a localized disturbance provides a measure of the degree of linkage of the associated vortex lines, and that helicity is conserved under the above conditions. A similar result had been proved in relation to the magnetic field in a perfectly conducting fluid by Woltjer (1956) and, indeed, it was through struggling to understand the physical meaning of Woltjer's result that I was led to think in terms of simple knots and linkages in convected vector fields generally.

Only recently has it come to my attention that the result concerning conservation of helicity was proved earlier by Moreau (1961) in a brief paper entitled 'Constantes d'un îlot tourbillonnaire en fluide parfait barotrope' in *Comptes rendus de l'Académie des Sciences*. It is clear also, from the following extract from a footnote to the paper, that Moreau appreciated the topological significance of his result: 'comme exemple d'îlot ayant un $\mathscr{H}$ non nul, nous proposons un système de deux anneaux tourbillonnaire de revolution, d'intensités respectives I_1 et I_2 *enlacés*, le tout plongé dans du fluide irrotationnel pour constituer un îlot D simplement connexe: on trouve alors $\mathscr{H} = 2I_1 I_2$'. Moreau used the letter $\mathscr{H}$ (as in Helmholtz) for the integral of $\mathbf{u}\cdot\boldsymbol{\omega}$; it is a good choice!

Indeed, as it turns out, it is an excellent choice because as pointed out by Kuznetsov & Mikhailov (1980) the invariant helicity is also identifiable with what is known to topologists as the Hopf invariant (Hopf 1931); it was shown by Whitehead (1947) that this invariant may be expressed as a volume integral which, as he said, 'in the notation of vector calculus' is none other than the integral of the scalar product of a vector field and its curl.

It seems to me that $\mathscr{H}$, being a quadratic invariant for a localized fluid motion (under the conditions defined above), has a status comparable with that of the kinetic energy associated with the disturbance. And yet kinetic energy appears to be much more fundamental, in that it can be defined for arbitrary dynamical systems, whereas helicity can be defined only for a continuum (since it involves the vorticity field, which can be defined only for a continuum). Here again, however, Moreau (1977) has made what appears to me to be an observation of great interest, *viz.* that conservation of helicity can be deduced by appropriate manipulation of Noether's theorem (for which, see Courant & Hilbert 1953, p. 262). Energy, momentum and angular momentum are likewise invariants which can be obtained by manipulation of Noether's theorem. Helicity thus appears to have status comparable with that of these classical invariants.

For two linked vortex tubes of strengths κ_1 and κ_2, and each of small cross-section, the helicity invariant reduces to $2n\kappa_1\kappa_2$, where

$$n = -\frac{1}{4\pi}\oint_{C_1}\oint_{C_2}\frac{\mathbf{x}_{12}\cdot(d\mathbf{x}_1\wedge d\mathbf{x}_2)}{|\mathbf{x}_{12}|^3}, \qquad (6.1)$$

where C_1 and C_2 are closed curves along the axes of the two tubes; $\mathbf{x}_1 \in C_1$, $\mathbf{x}_2 \in C_2$ and $\mathbf{x}_{12} = \mathbf{x}_1 - \mathbf{x}_2$. n is an integer – the winding number of C_1 with respect to C_2. It is of great interest to know whether there are any other topological invariants of any set of closed curves in three dimensions, analogous to (6.1), because one would then be able to reconstruct corresponding volume integral invariants analogous to helicity – and we have no proof as yet that such other invariants do not exist. The same question has been raised by Edwards (1967, 1968) not in the context of turbulence, but in the context of the statistical mechanics of polymers, for which long-chain molecules may be linked and knotted in a topologically invariant manner.

It is tempting to suppose (as did Edwards) that, for a single closed knotted curve C, the double integral analogous to (6.1), *viz.*

$$I = \frac{1}{4\pi}\oint_C \oint_C \frac{(\mathbf{x}-\mathbf{x}').(d\mathbf{x} \wedge d\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}, \tag{6.2}$$

should be a topological invariant; but, as shown by Vologodskii *et al.* (1974) (by explicit example), this is not in fact the case. This is somewhat surprising, since the integral (6.2) is certainly convergent, and it seems to be the degenerate form of the helicity invariant when the vorticity is confined to a single tube along C. Close inspection however shows that the helicity in this limit has an additional contribution from pairs of points $\mathbf{x}$, $\mathbf{x}'$ whose distance apart is comparable with the cross-sectional span of the tube; in fact, in the limit,

$$\kappa^{-2}\mathscr{H} = I + \frac{1}{2\pi}\oint_C \tau(s)\,ds, \tag{6.3}$$

where $\tau(s)$ is the *torsion* of C (a function of position s on C). As C is distorted, changes in I are compensated by changes in the torsion term so that $\mathscr{H}$ survives as an invariant. The invariant (6.3) appears to have been first obtained by Călugăreanu (1959).

Edwards (1967) also sought to construct an integral invariant characterizing the Borromean ring configuration – three rings, no two of which are linked, and yet which exhibit a 'triple' linkage. A vorticity field with this structure has zero helicity; is there then any other integral invariant that characterizes the (conserved) topological configuration? Edwards claimed to have found such an invariant (which involved integration round all three curves of the Borromean configuration) – but I was unable to convert his expression to a volume integral analogous to helicity; moreover, I am assured by topologists that it can be proved by homotopy theory that no 'classical' integral can possibly discriminate between the Borromean configuration and a configuration of three unlinked rings – the conclusion being that Edwards's claim must in fact be wrong.

There is, however, a small generalization of the helicity invariant that I would like to put on record. First, let $\mathbf{M} = \nabla \wedge \mathbf{N}$ and $\mathbf{P} = \nabla \wedge \mathbf{Q}$ be convected vector fields satisfying

$$\partial\mathbf{M}/\partial t = \nabla \wedge (\mathbf{u} \wedge \mathbf{M}), \quad \partial\mathbf{P}/\partial t = \nabla \wedge (\mathbf{u} \wedge \mathbf{P}), \tag{6.4}$$

and let S be a closed material surface with unit normal $\mathbf{n}$, on which $\mathbf{n}.\mathbf{M} = 0$ (permanently). Then it may easily be shown that

$$I_{MP} = \int_V \mathbf{M}.\mathbf{Q}\,dV \tag{6.5}$$

(the integral being over the volume V interior to S) is an invariant under arbitrary fluid motion $\mathbf{u}$. If we take $\mathbf{M} = \mathbf{P} = \mathbf{B}$ (magnetic field), then this is the Woltjer (1956) invariant; if $\mathbf{M} = \mathbf{P} = \boldsymbol{\omega}$, then it is the helicity invariant; if $\mathbf{M} = \mathbf{B}$ and $\mathbf{P} = \boldsymbol{\omega}$, it is the cross-helicity, also found by Woltjer (1956).

Now let $\mathbf{M} = \boldsymbol{\omega}$ and let $\mathbf{P}(\mathbf{x}, 0)$ be the field obtained from $\boldsymbol{\omega}(\mathbf{x}, 0)$ by the instantaneous volume-preserving distortion $\mathbf{x} \to \mathbf{X}(\mathbf{x})$, i.e.

$$P_i(\mathbf{X}, 0) = \omega_j(\mathbf{x}, 0)\, \partial X_i / \partial x_j. \tag{6.6}$$

$\mathbf{P}(\mathbf{x}, t)$ is then the field that evolves from $\mathbf{P}(\mathbf{x}, 0)$ according to (6.4). Then, for each $\mathbf{X}(\mathbf{x})$, (6.5) provides a quadratic functional of the vorticity field which is constant in time. Moreover this invariant represents (in some sense) the degree of linkage of the $\boldsymbol{\omega}$ field and the $\mathbf{P}$ field. It is my belief that invariants of this kind are the most general that can be obtained (in integral form), and that judicious choice of families of functions $\mathbf{X}(\mathbf{x})$ should permit complete discrimination between different topological structures of vortex lines. It seems to me that this is an area where interaction between topologists and fluid dynamicists could be rewarding.

The interest of helicity in the context of turbulence is of course that a new inviscid invariant implies some degree of constraint on the energy cascade process. Now, both energy *and* helicity are conserved by nonlinear interactions in the inertial range, and one must think in terms of a helicity cascade as well as an energy cascade, and of the possible coupling between these. Pioneering studies in this area have been carried out by Uriel Frisch and his group at the Observatoire de Nice in a series of papers (Frisch *et al.* 1975; André & Lesieur 1977; Pouquet & Patterson 1978), and by Kraichnan (1973, 1976*a*, *b*). The indications are that non-zero helicity in fact exerts only a mild restraining effect on the energy cascade process, that helicity itself tends to cascade (like a passive convected scalar) with a $k^{-\frac{5}{3}}$ spectrum, and that the relative helicity $|F(k)|/2kE(k)$ (where $F(k)$ is the helicity spectrum function) tends to decrease with increasing k.

What is really needed now is an experimental determination of the helicity spectrum in an experiment such as that of Ibbetson & Tritton (1975) in which the helicity is undoubtedly non-zero. This requires measurement of

$$\langle \mathbf{u}(\mathbf{x}) \,.\, \boldsymbol{\omega}(\mathbf{x}+\mathbf{r}) \rangle. \tag{6.7}$$

A first step towards this would be measurement of a correlation such as

$$\delta^{-1} \langle u_1(0,0,0)\,[u_3(r,\delta,0) - u_3(r,-\delta,0)] \rangle \tag{6.8}$$

for small δ; such a correlation is zero in reflexionally symmetric turbulence, and is non-zero only if circumstances are suitably contrived. A measurement of this kind would be of great interest – I issue this as an appeal (and a challenge) to experimental readers!

7. Postscript

The *JFM* time-scale may be 25 years, but 20 years appears to be a more appropriate unit for turbulence theory. Past landmarks of great significance, as discussed above, have been Taylor (1921), Kolmogorov (1941) and the Marseille meeting (1961) – and here we are now in 1981, amid a welter of new ideas injected partly from quantum field theory and statistical mechanics (e.g. the renormalization-group approach) and partly from pure mathematics (fractal dimension, strange attractors, development of singularities at finite time, etc.). Will the next 20 years bring a 'solution' to the problem of turbulence? I doubt it; but I hope that, by the year 2001, some at least of the ideas touched on in this article may have been more fully developed, and may lead to improved understanding of the effects of turbulence in circumstances where it really matters.

I have had some interesting discussions concerning the material of § 6 with Dr Brian Pollard (Bristol University) who drew my attention to the paper by Călugăreanu (1959). I would like to thank also Professor J.-J. Moreau, who sent me copies of his reprints which would otherwise have been inaccessible to me. Thanks also to Dr Y. Pomeau who explained homotopy theory to me (as it applies to the Borromean rings) in terms that I could understand.

REFERENCES

ALEMANY, A., MOREAU, R., SULEM, P.-L. & FRISCH, U. 1979 Influence of an external magnetic field on homogeneous MHD turbulence. *J. de Mécanique* **18**, 277.

ANDRÉ, J. D. & LESIEUR, M. 1977 Evolution of high Reynolds number isotropic three-dimensional turbulence; influence of helicity. *J. Fluid Mech.* **81**, 187.

BATCHELOR, G. K. 1950 On the spontaneous magnetic field in a conducting liquid in turbulent motion. *Proc. Roy. Soc.* A **201**, 405.

BATCHELOR, G. K. 1953 *The Theory of Homogeneous Turbulence*. Cambridge University Press.

BATCHELOR, G. K. 1959 Small scale variation of convected quantities like temperature in turbulent fluid, I. General discussion and the case of small conductivity. *J. Fluid Mech.* **5**, 113.

BATCHELOR, G. K. & PROUDMAN, I. 1956 The large-scale structure of homogeneous turbulence. *Phil. Trans. Roy. Soc.* A **248**, 369.

BATCHELOR, G. K. & TOWNSEND, A. A. 1956 Turbulent diffusion. In *Surveys in Mechanics* (ed. G. K. Batchelor & R. M. Davies), p. 352. Cambridge University Press.

BRISSAUD, A., FRISCH, U., LÉORAT, J., MAZURE, A., POUQUET, A., SADOURNY, R. & SULEM, P. L. 1973 Catastrophe énergétique et nature de la turbulence. *Ann. Geophys.* **29**, 539.

BRITTER, R. E., HUNT, J. C. R. & MUMFORD, J. C. 1979 The distortion of turbulence by a circular cylinder. *J. Fluid Mech.* **92**, 269.

CĂLUGĂREANU, G. 1959 L'Integrale de Gauss et l'Analyse des nœuds tridimensionelles. *Rev. Math. Pures Appl.* **4**, 5.

COURANT, R. & HILBERT, D. 1953 *Methods of Mathematical Physics*. Interscience.

DEARDORFF, J. W. 1970 A numerical study of three-dimensional turbulent channel flow at large Reynolds numbers. *J. Fluid Mech.* **41**, 453.

DEISSLER, R. G. 1961 Effects of inhomogeneity and of shear flow in weak turbulent fields. *Phys. Fluids* **4**, 1187.

EDWARDS, S. F. 1964 The statistical dynamics of homogeneous turbulence. *J. Fluid Mech.* **18**, 239.

EDWARDS, S. F. 1967 Statistical mechanics with topological constraints: I. *Proc. Phys. Soc.* **91**, 513.

EDWARDS, S. F. 1968 Statistical mechanics with topological constraints: II. *J. Phys.* A (*Proc. Phys. Soc.*) ser. 2, **1**, 15.

ELSASSER, W. M. 1946 Induction effects in terrestrial magnetism, I. Theory. *Phys. Rev.* **69**, 106.

FERMIGIER, M. 1980 Mésures sur les écoulements laminaires et turbulents par marquage optique. Thèse du 3^{me} cycle, Université Paris VI.

FORSTER, D., NELSON, D. R. & STEPHEN, M. J. 1977 Large distance and long-time properties of a randomly stirred fluid. *Phys. Rev.* A **16**, 732.

FOURNIER, J. D. 1977 Quelques méthodes systématiques de développement en turbulence homogène. Thèse de 3ème cycle, Université de Nice.

FRISCH, U., POUQUET, A., LÉORAT, J. & MAZURE, A. 1975 Possibility of an inverse cascade of magnetic helicity in magnetohydrodynamic turbulence. *J. Fluid Mech.* **68**, 769.

FRISCH, U., SULEM, P.-L. & NELKIN, M. 1978 A simple dynamical model of intermittent fully developed turbulence. *J. Fluid Mech.* **87**, 719.

GRANT, H. L. 1958 The large eddies of turbulent motion. *J. Fluid Mech.* **4**, 149.

GRANT, H. L., STEWART, R. W. & MOILLIET, A. 1962 Turbulence spectra from a tidal channel. *J. Fluid Mech.* **12**, 241.

GURVICH, A. S. 1960 Experimental investigation of frequency spectra of the vertical wind velocity in the atmospheric surface layer. *Dokl. Akad. Nauk S.S.S.R.* **132**, 806.

HEISENBERG, W. 1948 On the theory of statistical and isotropic turbulence. *Proc. Roy. Soc.* A **195**, 402.

HOPF, H. 1931 Über die Abbildungen der dreidimensionalen Sphäre auf die Kugelfläche. *Math. Annalen* **104**, 637.

HOWELLS, I. D. 1960 An approximate equation for the spectrum of a conserved scalar quantity in a turbulent fluid. *J. Fluid Mech.* **9**, 104.

HUNT, J. R. C. 1973 A theory of turbulent flow round two-dimensional bluff bodies. *J. Fluid Mech.* **61**, 625.

HUNT, J. C. R. 1978 A review of the theory of rapidly distorted turbulent flows and its applications. *Fluid Dynamics Trans.* **9**, 121.

HUNT, J. C. R. & GRAHAM, J. M. R. 1977 Free-stream turbulence near plane boundaries. *J. Fluid Mech.* **84**, 209.

IBBETSON, A. & TRITTON, D. J. 1975 Experiments on turbulence in a rotating fluid. *J. Fluid Mech.* **68**, 639.

KELVIN, LORD (then W. THOMSON) 1868 *Trans. Roy. Soc. Edin.* **25**, 217.

KOLMOGOROV, A. N. 1941 The local structure of turbulence in incompressible viscous fluid for very large Reynolds numbers. *Dokl. Akad. Nauk S.S.S.R.* **30**, 299.

KOLMOGOROV, A. N. 1949 On the disintegration of drops in turbulent flow. *Dokl. Akad. Nauk S.S.S.R.* **66**, 825.

KOLMOGOROV, A. N. 1962 A refinement of previous hypotheses concerning the local structure of turbulence in a viscous incompressible fluid at high Reynolds number. *J. Fluid Mech.* **13**, 82.

KRAICHNAN, R. H. 1959 The structure of isotropic turbulence at very high Reynolds numbers. *J. Fluid Mech.* **5**, 497.

KRAICHNAN, R. H. 1965 Lagrangian–history closure approximation for turbulence. *Phys. Fluids* **8**, 575.

KRAICHNAN, R. H. 1973 Helical turbulence and absolute equilibrium. *J. Fluid Mech.* **59**, 745.

KRAICHNAN, R. H. 1976*a* Diffusion of weak magnetic fields by isotropic turbulence. *J. Fluid Mech.* **75**, 657.

KRAICHNAN, R. H. 1976*b* Diffusion of passive-scalar and magnetic fields by helical turbulence. *J. Fluid Mech.* **77**, 753.

KRAICHNAN, R. H. 1977 Eulerian and Lagrangian renormalization in turbulence theory. *J. Fluid Mech.* **83**, 349.

KRAICHNAN, R. H. 1978 Negative diffusivity. *Woods Hole Oceanographic Inst. Tech. Rep.* WHO I-78-67, 136.

KRAUSE, F. & RÄDLER, K.-H. 1981 *Mean Field Magnetohydrodynamics and Dynamo Theory.* Pergamon.

KUZNETSOV, E. A. & MIKHAILOV, A. V. 1980 On topological meaning of canonical Clebsch variables. *Phys. Lett.* **77**A, 37.

LANDAHL, M. T. 1967 A wave-guide model for turbulent shear flow. *J. Fluid Mech.* **29**, 441.

MANDELBROT, B. B. 1974 Intermittent turbulence in self-similar cascades: divergence of high moments and dimension of the carrier. *J. Fluid Mech.* **62**, 331.

MANDELBROT, B. B. 1975 On the geometry of homogeneous turbulence, with stress on the fractal dimension of the iso-surfaces of scalars. *J. Fluid Mech.* **72**, 401.

MOFFATT, H. K. 1967*a* The interaction of turbulence with strong wind shear. In *Atmospheric Turbulence and Radio Wave Propagation* (ed. A. M. Yaglom & V. I. Tatarski), p. 139. Moscow: Nauka.

MOFFATT, H. K. 1967*b* On the suppression of turbulence by a uniform magnetic field. *J. Fluid Mech.* **28**, 571.

MOFFATT, H. K. 1969 The degree of knottedness of tangled vortex lines. *J. Fluid Mech.* **35**, 117.

MOFFATT, H. K. 1978 *Magnetic Field Generation in Electrically Conducting Fluids*. Cambridge University Press.

MONIN, A. S. & YAGLOM, A. M. 1971 *Statistical Fluid Mechanics*, vol. I. Massachusetts Institute of Technology Press.

MONIN, A. S. & YAGLOM, A. M. 1975 *Statistical Fluid Mechanics*, vol. II. Massachusetts Institute of Technology Press.

MOREAU, J. J. 1961 Constantes d'un îlot tourbillonnaire en fluid parfait barotrope. *C. R. Akad. Sci. Paris* **252**, 2810.

MOREAU, J. J. 1977 Sur les intégrales premières de la dynamique d'un fluid parfait barotrope et le théorème de Helmholtz–Kelvin. *Séminaire d'Analyse Convexe, Montpellier, Exposé* no. 7.

NELKIN, M. 1974 Turbulence, critical phenomena and intermittency. *Phys. Rev.* A **9**, 388.

NELKIN, M. 1975 Scaling theory of hydrodynamic turbulence. *Phys. Rev.* A **11**, 1737.

NOVIKOV, E. A. 1961 Energy spectrum of turbulent flow of an incompressible fluid. *Dokl. Akad. Nauk S.S.S.R.* **139**, 331.

OBOUKHOV, A. M. 1962 Some specific features of atmospheric turbulence. *J. Fluid Mech.* **13**, 77.

PARKER, E. N. 1971 The generation of magnetic fields in astrophysical bodies, III. Turbulent diffusion of fields and efficient dynamos. *Astrophys. J.* **163**, 279.

PARKER, E. N. 1979 *Cosmical Magnetic Fields*. Clarendon.

PATTERSON, G. S. & CORRSIN, S. 1966 Computer experiments on a random walk with both Eulerian and Lagrangian statistics. In *Dynamics of Fluids and Plasmas* (ed. S. I. Pai *et al.*), p. 275. Academic.

PEARSON, J. R. A. 1959 The effect of uniform distortion on weak homogeneous turbulence. *J. Fluid Mech.* **5**, 274.

PHYTHIAN, R. & CURTIS, W. D. 1978 The effective long-time diffusivity for a passive scalar in a Gaussian model fluid flow. *J. Fluid Mech.* **89**, 241.

POUQUET, A., FOURNIER, J. D. & SULEM, P.-L. 1978 Is helicity relevant for large scale steady state three-dimensional turbulence? *J. Physique Lettres* **39**, 199.

POUQUET, A., FRISCH, U. & LÉORAT, J. 1976 Strong MHD turbulence and the nonlinear dynamo effect. *J. Fluid Mech.* **77**, 321.

POUQUET, A. & PATTERSON, G. S. 1978 Numerical simulation of helical magnetohydrodynamic turbulence. *J. Fluid Mech.* **85**, 305.

ROBERTS, P. H. & STIX, M. 1971 The turbulent dynamo: a translation of a series of papers by F. Krause, K.-H. Rädler & M. Steenbeck. NCAR, Boulder, Colorado, *Tech. Note* 60.

ROSE, H. A. 1977 Eddy diffusivity, eddy noise and subgridscale modelling. *J. Fluid Mech.* **81**, 719.

ROSE, H. A. & SULEM, P. L. 1978 Fully developed turbulence and statistical mechanics. *J. de Phys.* **39**, 441.

SAFFMAN, P. G. 1960 On the effect of molecular diffusivity in turbulent diffusion. *J. Fluid Mech.* **8**, 273.

SAFFMAN, P. G. 1962 Some aspects of the effects of the molecular diffusivity in turbulent diffusion. In *Mécanique de la Turbulence*. Ed. du CNRS, no. 108. Paris.

SAFFMAN, P. G. 1967 The large-scale structure of homogeneous turbulence. *J. Fluid Mech.* **27**, 581.

SIGGIA, E. D. & PATTERSON, G. S. 1978 Intermittency effects in a numerical simulation of stationary three-dimensional turbulence. *J. Fluid Mech.* **86**, 567.

SCHEFFER, V. 1976 Turbulence and Hausdorff dimension. In *Turbulence and Navier–Stokes Equation*. Lecture Notes in Mathematics, no. 565 (ed. A. Dold & B. Eckmann). Springer.

SNYDER, W. H. & LUMLEY, J. L. 1971 Some measurements of particle velocity autocorrelation functions in turbulent flow. *J. Fluid Mech.* **48**, 41.

STEENBECK, M., KRAUSE, F. & RÄDLER, K.-H. 1966 A calculation of mean electromotive force in an electrically conducting fluid in turbulent motion, under the influence of Coriolis forces. *Z. Naturforsch.* **21** *a*, 369.

STEENBECK, M. & KRAUSE, F. 1966 The generation of stellar and planetary magnetic fields by turbulent dynamo action. *Z. Naturforsch.* **21** *a*, 1285.

TAYLOR, G. I. 1921 Diffusion by continuous movements. *Proc. Lond. Math. Soc.* A **20**, 196.

TAYLOR, G. I. 1935 Statistical theory of turbulence. Parts 1–4. *Proc. Roy. Soc.* A **151**, 421.

TOWNSEND, A. A. 1951 On the fine-scale structure of turbulence. *Proc. Roy. Soc.* A **208**, 534.

TOWNSEND, A. A. 1956 *The Structure of Turbulent Shear Flow*. Cambridge University Press.

TOWNSEND, A. A. 1970 Entrainment and the structure of turbulent flow. *J. Fluid Mech.* **41**, 13.

TOWNSEND, A. A. 1976 *The Structure of Turbulent Shear Flow*, 2nd edn. Cambridge University Press.

VOLOGODSKII, A. V., LUKASHIN, A. V., FRANK-KAMENTSKII, M. D. & ANSHELEVICH, V. V. 1974 The knot problem in statistical mechanics of polymer chains. *Sov. Phys. J. Exp. Theor. Phys.* **39**, 1059.

WHITEHEAD, J. H. C. 1947 An expression of Hopf's invariant in an integral. *Proc. Nat. Acad. Sci. USA* **33**, 117.

WOLTJER, L. 1956 Astrophysical evidence for strong magnetic fields. *Ann. New York Acad. Sci.* **257**, 76.